TUTORIAL-6

1. The two lines of regression are $3x+2y=26$, $6x+y=31$. Find $(\bar{x},\bar{y})$, r_{xy} and where $\sigma x=1$. find σy.

Sol: Since the regression lines always passes through $(\bar{x},\bar{y})$.

$$3\bar{x}+2\bar{y}=26$$
$$6\bar{x}+\bar{y}=31$$

on solving $\bar{x}=4$, $\bar{y}=7$

Let us assume $3\bar{x}+2\bar{y}=26$ is the regression of y on x.

$$2\bar{y}=-3\bar{x}+26$$
$$\bar{y}=-\frac{3}{2}\bar{x}+\frac{26}{2}$$
$$\Rightarrow byx=\frac{-3}{2}$$

Let us assume $6\bar{x}+\bar{y}=31$ is the regression of x on y

$$6\bar{x}=-\bar{y}+31$$
$$\bar{x}=-\frac{1}{6}\bar{y}+\frac{31}{6}$$
$$\Rightarrow bxy=\frac{-1}{6}$$

$$r_{xy}=\pm\sqrt{bxy \times byx}$$
$$=\pm\sqrt{\frac{-1}{6}\times\frac{-3}{2}}$$
$$=\pm\sqrt{\frac{3}{12}}=\pm\sqrt{\frac{1}{4}}=\pm\sqrt{0.25}=\pm 0.5$$
$$r_{xy}=-0.5$$

$\sigma x=1$, $\sigma y=?$

W.K.T. $bxy = r\frac{\sigma x}{\sigma y}$

$$\frac{-1}{6}=-0.5\times\frac{1}{\sigma y}$$
$$\frac{1}{6}=\frac{0.5}{\sigma y}$$
$$\sigma y=6\times 0.5$$
$$\sigma y=3$$

2. Prove that Correlation coefficient lies between $(-1, 1)$.

i.e. T.P $-1 \leq r_{xy} \leq 1$

Proof:

Let $E[a(x-\bar{x}) + (y-\bar{y})]^2 \geq 0$

$$a^2 E[x-\bar{x}]^2 + E(y-\bar{y})^2 + 2aE(x-\bar{x})(y-\bar{y}) \geq 0$$

$$a^2 \sigma x^2 + \sigma y^2 + 2a \, Cov(x,y) \geq 0$$

The above eqn ≥ 0 only when $B^2 - 4AC \leq 0$

ie, $[2\,Cov(x,y)]^2 - 4\sigma x^2 \sigma y^2 \leq 0$

$$4[Cov(x,y)]^2 \leq 4 \sigma x^2 \sigma y^2$$

$$Cov^2 xy \leq \sigma x^2 \sigma y^2$$

$$\frac{Cov^2 xy}{\sigma x^2 \sigma y^2} \leq 1$$

$$r^2 xy \leq 1$$

$$|r_{xy}| \leq 1$$

$$-1 \leq r_{xy} \leq 1$$

Hence proved.

3. Obtain the lines of regression from the following data, also estimate the value of Y when $X = 38$ and value of X when $Y = 18$.

X :	22	26	29	30	31	31	34	35
Y :	20	20	21	29	27	24	27	31

Sol:

$$\bar{X} = \frac{\Sigma x}{n} = \frac{238}{8} = 29.75 = E(x)$$

$$\bar{Y} = \frac{\Sigma y}{n} = \frac{199}{8} = 24.875 = E(Y) \qquad E(XY) = 752.875$$

$$\sigma x = \sqrt{E(x^2) - (E(x))^2} = \sqrt{900.5 - (29.75)^2}$$
$$= \sqrt{900.5 - 885.0625} = \sqrt{15.4375} = 3.929$$

$$\sigma y = \sqrt{E(Y^2) - (E(Y))^2} = \sqrt{634.625 - (24.875)^2}$$
$$= \sqrt{634.625 - 618.756} = \sqrt{15.869} = 3.983$$

$$r_{xy} = \frac{E(xy) - E(x)E(y)}{\sigma x \sigma y}$$

$$= \frac{(752.875) - (29.75)(24.875)}{(3.929)(3.983)}$$

$$= \frac{752.875 - 740.031}{15.808}$$

$$= \frac{12.844}{15.808}$$

$$r_{xy} = 0.8125$$

Regression of X on Y is

$$X - \bar{X} = r \frac{\sigma x}{\sigma y}(Y - \bar{Y})$$

$$X - 29.75 = (0.8125)\frac{3.929}{3.983}(Y - 24.875)$$

$$X - 29.75 = 0.8015 (Y - 24.875)$$

$$X - 29.75 = 0.8015 Y - 19.937$$

$$X - 0.8015 Y = 29.75 - 19.937$$

$$X - 0.8015 Y = 9.813 \longrightarrow ①$$

Regression of Y on X is

$$Y - \bar{Y} = r \frac{\sigma y}{\sigma x}(X - \bar{X})$$

$$Y - 24.875 = (0.8125)\frac{3.983}{3.929}(X - 29.75)$$

$$X - 24.875 = (0.8125)(1.013)(X - 29.75)$$

$$X - 24.875 = (0.823)(X - 29.75)$$

$$Y - 24.875 = 0.823 X - 24.484$$

$$X - 0.823 X = 24.875 - 24.484 = 0.391$$

$$Y - 0.823 X = 0.391 \longrightarrow ②$$

To find X when Y = 18

put Y = 18 in ①

$$X - 0.8015(18) = 9.813$$

$$X - 14.427 = 9.813$$

$$X = 9.813 + 14.427$$

$$\boxed{X = 24.24}$$

To find Y when X = 38

Put X = 38 in ②

$$Y - 0.823 X = 0.391$$

$$Y - 0.823(38) = 0.391$$

$$Y - 31.274 = 0.391$$

$$Y = 0.391 + 31.274$$

$$\boxed{Y = 31.665}$$

4. Given that X and Y are independent random variables which is given by. $f(x) = \begin{cases} 4ax, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$, $f(y) = \begin{cases} 4by, & 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

Prove that. $u = X+Y$, $V = X-Y$ are uncorrelated.

Sol: u and v are uncorrelated $\Rightarrow Cov(u,v) = 0$.

$$\text{Let } Cov(u,v) = E(uv) - E(u)E(v)$$
$$= E[(X+Y)(X-Y)] - E(X+Y)(X-Y)$$
$$= E(X^2-Y^2) - [E(x)+E(Y)][E(X)-E(Y)]$$
$$= E(X^2-Y^2) - [(E(X))^2 - (E(Y))^2]$$

To find a

w.k.t $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 4ax\,dx = 1$$
$$4a\left(\frac{x^2}{2}\right)_0^1 = 1$$
$$2a(1-0) = 1$$
$$2a = 1$$
$$a = 1/2$$

To find b.

$$\int_{-\infty}^{\infty} f(y)\,dy = 1$$
$$\int_0^1 4by\,dy = 1$$
$$4b\left(\frac{y^2}{2}\right) = 1$$
$$2b(1-0) = 1$$
$$2b = 1$$
$$b = 1/2$$

$$f(x) = \begin{cases} 4\left(\frac{1}{2}\right)x & , 0<x<1 \\ 0 & , \text{otherwise} \end{cases} \Rightarrow f(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

$$f(y) = \begin{cases} 4\left(\frac{1}{2}\right)y & , 0<y<1 \\ 0 & , \text{otherwise} \end{cases} \Rightarrow f(y) = \begin{cases} 2y, & 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Let } E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(2x)\,dx = 2\int_0^1 x^2\,dx = 2\left(\frac{x^3}{3}\right)_0^1$$

$$E(x) = \frac{2}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2(2x)\,dx = 2\int_0^1 x^3\,dx = 2\left(\frac{x^4}{4}\right)_0^1$$

$$E(X) = \frac{1}{2}$$

Let $E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y(2y)\,dy = 2\left(\frac{y^3}{3}\right)_0^1$

$$E(Y) = \frac{2}{3}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_0^1 y^2(2y)\,dy = 2\left(\frac{y^4}{4}\right)_0^1$$

$$E(Y^2) = \frac{1}{2}$$

$$\therefore \operatorname{Cov}(u,v) = \left(\frac{1}{2} - \frac{1}{2}\right) - \left[\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2\right]$$

$$= 0.$$

$$r_{uv} = \frac{\operatorname{Cov}(u,v)}{\sigma_x \sigma_y} = 0$$

$$r_{uv} = 0$$

Hence u & V are uncorrelated.

5. Find the angle between two lines of regression.

W.K.T $\quad \theta = \tan^{-1}\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$

Target equation of the straight line is

$$(y - y_1) = \frac{-1}{m}(x - x_1)$$

Let the regression of Y on X is

$$Y - \bar{Y} = r\frac{\sigma_y}{\sigma_x}(X - \bar{X})$$

$$\frac{-1}{m} = r\frac{\sigma_y}{\sigma_x}$$

$$\Rightarrow m_1 = -\frac{\sigma_x}{r\sigma_y}$$

Let the regression of X on Y is

$$X - \bar{X} = r\frac{\sigma_x}{\sigma_y}(Y - \bar{Y})$$

$$\Rightarrow Y - \bar{Y} = \frac{\sigma_y}{r\sigma_x}(X - \bar{X})$$

$$\Rightarrow \frac{-1}{m} = \frac{\sigma y}{r \sigma x}$$

$$\Rightarrow m_2 = \frac{-r\sigma x}{\sigma y}.$$

$$\theta = \tan^{-1}\left[\frac{-\frac{r\sigma x}{\sigma y} + \frac{\sigma x}{r \sigma y}}{1 + \frac{\sigma x}{r\sigma y} \cdot \frac{r \sigma x}{\sigma y}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{-r^2\sigma x + \sigma x}{r\sigma y}}{\frac{\sigma y^2 + \sigma x^2}{\sigma y^2}}\right]$$

$$= \tan^{-1}\left[\frac{\sigma x(1-r^2)\sigma y}{r(\sigma x^2 + \sigma y^2)}\right]$$

$$\therefore \theta = \tan^{-1}\left[\left(\frac{1-r^2}{r}\right)\frac{\sigma x \sigma y}{\sigma x^2 + \sigma y^2}\right]$$

My